Embedding first countable spaces into first countable countably compact spaces

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Various set-theoretic hypotheses allow us to embed first countable regular Hausdorff spaces into countably compact ones. The following theorem is definitive for a special class of them.

Theorem 1. *The set-theoretic axiom* $\mathfrak{b} = \mathfrak{c}$ *is equivalent to the statement that every locally compact, first countable Hausdorff space can be embedded into a locally compact, first countable, countably compact Hausdorff space.*

In the absence of local compactness, we seem to need another small uncountable cardinal, the splitting number $\mathfrak{s}$, to be brought into play. The following general theorem is a starting point.

Theorem 2. $[\mathfrak{b} = \mathfrak{s} = \mathfrak{c}]$ *Every regular, first countable space* X *of weight* $< \mathfrak{c}$ *can be embedded into a regular, first countable, countably compact space. If* X *is 0-dimensional, the extension can be made 0-dimensional as well.*

The following is immediate from the elementary facts that weight $\leq$ cardinality for first countable spaces and that every Tychonoff space X of cardinality $< \mathfrak{c}$ is 0-dimensional.

Corollary. $[\mathfrak{b} = \mathfrak{s} = \mathfrak{c}]$ *Every first countable Tychonoff space* X *of cardinality* $< \mathfrak{c}$ *can be embedded into a Tychonoff, first countable, countably compact space.*

To replace cardinality with weight for Tychonoff spaces has thus far required a stronger accompaniment to $\mathfrak{b} = \mathfrak{c}$ than $\mathfrak{s} = \mathfrak{c}$. The stronger axiom is that there is a simple P-point on ω of cofinality